On conserved densities and boundary conditions for the Davey–Stewartson equations

2010 J. Phys. A: Math. Theor. 43 045206

(http://iopscience.iop.org/1751-8121/43/4/045206)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 045206 (13pp) doi:10.1088/1751-8113/43/4/045206

On conserved densities and boundary conditions for the Davey–Stewartson equations

V Rosenhaus[1] **and M Gandarias**[2]

[1] Department of Mathematics and Statistics, California State University, Chico, CA 95929, USA
[2] Department of Mathematics, University of Cadiz, 11510 Puerto Real, Cadiz, Spain

E-mail: vrosenhaus@csuchico.edu and marialuz.gandarias@uca.es

Received 17 September 2009, in final form 27 October 2009
Published 4 January 2010
Online at stacks.iop.org/JPhysA/43/045206

Abstract
In the paper we discuss local conservation laws with non-vanishing conserved densities and corresponding boundary conditions for the Davey–Stewartson equations. Proceeding with an infinite symmetry algebra of the Davey–Stewartson system we generate a finite number of conserved quantities through appropriate asymptotic conditions.

PACS numbers: 02.20.Tw, 11.30.−j

1. Introduction

In the present paper we will discuss conservation laws for the system of Davey–Stewartson (DS) equations corresponding to its infinite classical Lie point symmetry group. The DS system was originally formulated as equations of evolution of weakly nonlinear water waves in 2+1 dimensions (Davey and Stewartson 1974). It describes the nonlinear (resonant) interaction between long and short waves in the case of shallow water in 2+1 dimensions (Djordjevic and Redecopp 1977); see also Benney and Roskes (1969) and Kaup (1993) and a discussion therein. The DS equations arise in plasma physics and nonlinear optics, e.g. Sulem and Sulem (1999). The DS system can be considered as a natural two-dimensional generalization of a nonlinear Schrödinger equation.

Many aspects of the DS equations were extensively studied. It has been shown that the DS equations can be solved by the inverse scattering method (Anker and Freeman 1978, Fokas and Ablowitz 1984). A number of soliton-type solutions of the DS equations have been found (solitons, lump solitons, ripplons, dromions, etc), see e.g. Zakharov and Shabat (1973), Nakamura (1982), Boiti *et al* (1988) and Fokas and Santini (1990).

The Lie point symmetry group for the DS equations was calculated in Champagne and Winternitz (1988) and was shown to correspond to an infinite Lie algebra involving six arbitrary functions of time. The connection of this infinite-dimensional symmetry algebra to gauge transformations of the Schrödinger equation and corresponding infinite number of conservation laws (continuity equations) were discussed in Omote (1988). The existence of an infinite number of conservation laws for the DS system was discussed in Kulish and Lipovsky (1987) and Fokas and Santini (1988a, 1988b). Conservation laws for the quantized DSI system were studied in Pang and Zhao (1993). Formal series symmetries of the DS equations were discussed in Lou and Hu (1994).

The goal of the present paper is to find essential conservation laws (Rosenhaus 2003) for the DS system that are associated with its infinite Lie point symmetry group. Out of the infinite set we will be looking for those continuity equations that lead to non-vanishing conserved densities (essential conservation laws). For each essential conservation law of the DS equations we will identify certain boundary conditions that make possible the existence of a corresponding conserved quantity. In our derivation we will follow the approach developed in Rosenhaus (2002).

The relationship between variational symmetries and conservation laws has a long history and goes back to the classic Noether results (Noether 1918), see also Olver (1986). According to the second Noether theorem (Noether 1918) infinite variational symmetries with arbitrary functions of all independent variables do not lead to conservation laws but to a certain relation between equations of the original differential system. Infinite variational symmetries with arbitrary functions of not all independent variables were shown to lead to a finite number of essential local conservation laws (Rosenhaus 2002). For infinite symmetries containing arbitrary functions of *t,* it was shown in Rosenhaus (2002) that the main factor determining the existence of corresponding conservation laws is the form of boundary conditions, see also Rosenhaus (2003, 2005, 2006).

2. Infinite symmetries and essential conservation laws

By a conservation law for a differential system

$$\omega^a(x, u, u_i, \dots) = 0, \qquad i = 1, \dots, m+1, \qquad a = 1, \dots, n, \quad u_i^a \equiv \partial u^a/\partial x^i$$

is meant a continuity equation

$$D_i K_i \doteq 0, \qquad K_i = K_i(x, u, u_j \dots), \quad i, j = 1, \dots, m+1, \quad x^i = (x^1, x^2, \dots, x^m, t)$$

(K_i are smooth functions), which is satisfied for any solutions of the original system (Olver 1986). Each conservation law is defined up to an equivalence transformation $K_i \to K_i + P_i$, where $D_i P_i \doteq 0$. Two conservation laws belong to the same equivalence class if they differ by a trivial conservation law. For trivial conservation laws the components of the vector K_i vanish on the solutions: $K_i \doteq 0\,(i = 1, \dots, m+1)$ or the continuity equation is satisfied in the whole space: $D_i K_i = 0$ (Olver 1986). By an *essential* conservation law (Rosenhaus 2003), we mean such a non-trivial conservation law $D_i K_i \doteq 0$, which gives rise to a non-vanishing conserved density:

$$D_t \int_D K_t \, dx^1 \, dx^2 \cdots dx^m \doteq 0, \qquad x \in D \subset R^{m+1}, \qquad K_t \,\dot{\neq}\, 0. \tag{1}$$

We consider the functions $u = u(x)$ defined on a region D of $(m+1)$-dimensional spacetime. Let

$$S = \int_D L\big(x^i, u^a, u_i^a, \dots\big) \, d^{m+1}x \qquad a = 1, \dots, n, \qquad i, j = 1, \dots, m+1$$

be the action functional, where L is the Lagrangian density. The equations of motion are

$$E^a(L) \equiv \omega^a(x, u, u_i, u_{ij}, \dots) = 0, \qquad a = 1, \dots, n, \qquad i, j = 1, \dots, m+1, \tag{2}$$

where E is the Euler–Lagrange operator:

$$E^a = \frac{\partial}{\partial u^a} - \sum_i D_i \frac{\partial}{\partial u_i^a} + \sum_{i\leqslant j} D_i D_j \frac{\partial}{\partial u_{ij}^a} + \cdots. \tag{3}$$

Consider an infinitesimal transformation with the canonical operator:

$$X_\alpha = \alpha^a \frac{\partial}{\partial u^a} + (D_i\alpha^a)\frac{\partial}{\partial u_i^a} + \sum_{i\leqslant j}(D_i D_j\alpha^a)\frac{\partial}{\partial u_{ij}^a} + \cdots, \tag{4}$$

$$\alpha^a = \alpha^a(x, u, u_i, \ldots) \qquad i, j = 1, \ldots, m+1, \qquad a = 1, \ldots, n$$

(summation over repeated indices is assumed). Variation of the functional S under the transformation with operator X_α is

$$\delta S = \int_D X_\alpha L \, \mathrm{d}^{m+1}x. \tag{5}$$

X_α is a variational (Noether) symmetry if

$$X_\alpha L = D_i M_i, \qquad M_i = M_i(x, u, u_i, \ldots), \qquad i = 1, \ldots, m+1, \tag{6}$$

where M_i are smooth functions. In the future we will use the Noether identity (Rosen 1972) (see also e.g. Ibragimov (1985) or Rosenhaus (2002)):

$$X_\alpha = \alpha^a E^a + D_i R_{\alpha i}, \qquad i = 1, \ldots, m+1, \qquad a = 1, \ldots, n, \tag{7}$$

$$R_{\alpha i} = \alpha^a \frac{\partial}{\partial u_i^a} + \left\{ \sum_{k\geqslant i}(D_k\alpha^a) - \alpha^a \sum_{k\leqslant i} D_k \right\} \frac{\partial}{\partial u_{ik}^a} + \cdots. \tag{8}$$

Applying the Noether identity (7) (with (8)) to L and combining with (6) we obtain

$$D_i(M_i - R_{\alpha i}L) = \alpha^a\omega^a, \qquad i = 1, \ldots, m+1, \qquad a = 1, \ldots, n. \tag{9}$$

Equation (9) applied to the solution manifold ($\omega = 0, D_i\omega = 0, \ldots$) leads to a continuity equation

$$D_i(M_i - R_{\alpha i}L) \doteq 0, \qquad i = 1, \ldots, m+1. \tag{10}$$

Thus, any one-parameter variational symmetry transformation α (6) leads to a conservation law (10) (the first Noether theorem) with the characteristic α. The second Noether theorem (Noether 1918) deals with a case of an infinite variational symmetry group where the symmetry vector α is of the form

$$\alpha^a = a^a p(x) + b_i^a D_i p(x) + c_{ij}^a D_i D_j p(x) + \cdots, \qquad a = 1, \ldots, n, \tag{11}$$

and $p(x)$ is an arbitrary function of all base variables of the space. Unlike with the first Noether theorem, a consequence of an infinite symmetry (11) of functional S is not a conservation law but a certain relation between the original differential equations (Noether 1918). A general situation when $p(x)$ is an arbitrary function of not all base variables was analyzed in Rosenhaus (2002). For a Noether symmetry transformation X_α we have

$$\delta S = \int_D \delta L \, \mathrm{d}^{m+1}x = \int_D X_\alpha L \, \mathrm{d}^{m+1}x = \int_D D_i M_i \, \mathrm{d}^{m+1}x = 0,$$

$$x \in D \subset R^{m+1}. \tag{12}$$

Therefore, the following conditions for M_i (Noether boundary conditions) should hold (Rosenhaus 2002):

$$M_i(x, u, \ldots)|_{x^i \to \partial D} = 0, \qquad \forall i = 1, \ldots, m+1. \tag{13}$$

Equations (13) are usually satisfied for a 'regular' asymptotic behavior, $u, u_i \rightarrow 0$ as $x \rightarrow \pm\infty$, or for periodic solutions. Let us consider now another type of boundary conditions related to the existence of local conserved quantities. Integrating equation (10) over the space $(x^1, x^2, \ldots, x^m)$ we get

$$\int \mathrm{d}x^1\, \mathrm{d}x^2 \cdots \mathrm{d}x^m D_t(M_t - R_{\alpha t}L) \doteq \int \mathrm{d}x^1 \cdots \mathrm{d}x^m \sum_{i=1}^{m} D_i(R_{\alpha i}L - M_i). \tag{14}$$

Applying the Noether boundary condition (13) and requiring the lhs of (14) to vanish on the solution manifold we obtain the 'strict' boundary conditions (Rosenhaus 2002)

$$R_{\alpha 1}L|_{x^1 \to \partial D} = R_{\alpha 2}L|_{x^2 \to \partial D} = \cdots = R_{\alpha m}L|_{x^m \to \partial D} = 0. \tag{15}$$

In this paper, we will be mainly interested in symmetries with arbitrary functions of time $\gamma(t)$. It is easy to demonstrate that infinite symmetries with arbitrary functions of t can lead only to a finite number of essential conservation laws for equations with first-order Lagrangian functions, $L = L(u, u_x, u_t)$; for details and a generalization to higher order Lagrangians see Rosenhaus (2002). Consider the variational symmetry α of the form

$$\alpha^a = a^a\gamma(t) + b^a\gamma'(t) + c^a\gamma''(t) + \cdots + h^a\gamma^{(k)}(t), \qquad a = 1, \ldots, n. \tag{16}$$

In order for a differential system to possess Noether local conserved quantities, both Noether (13) and strict boundary conditions (15) have to be satisfied. The corresponding Noether conservation law can be found in the form

$$D_t \int \mathrm{d}x^1\, \mathrm{d}x^2 \cdots \mathrm{d}x^m\, (M_t - R_{\alpha t}L) \doteq 0. \tag{17}$$

Writing M_t as

$$M_t = A\gamma(t) + B\gamma'(t) + C\gamma''(t) + \cdots + H\gamma^{(l)}(t), \tag{18}$$

from (17), we obtain

$$D_t \int \mathrm{d}x^1\, \mathrm{d}x^2 \cdots \mathrm{d}x^m [\gamma(t)A_1 + \gamma'(t)A_2 + \cdots + \gamma^{(l)}(t)A_l] \doteq 0, \tag{19}$$

where

$$A_1 = \left(A - a\frac{\partial L}{\partial u_t}\right), \qquad A_2 = \left(B - b\frac{\partial L}{\partial u_t}\right), \ldots, \qquad A_l = \left(H - h\frac{\partial L}{\partial u_t}\right).$$

Since $\gamma(t)$ is arbitrary we get

$$\int \mathrm{d}x^1\, \mathrm{d}x^2 \cdots \mathrm{d}x^m \left(A - a\frac{\partial L}{\partial u_t}\right) \doteq \int \mathrm{d}x^1\, \mathrm{d}x^2 \cdots \mathrm{d}x^m \left(B - b\frac{\partial L}{\partial u_t}\right) \doteq \cdots$$
$$\doteq \int \mathrm{d}x^1\, \mathrm{d}x^2 \cdots \mathrm{d}x^m \left(H - h\frac{\partial L}{\partial u_t}\right) \doteq 0. \tag{20}$$

Obviously, equation (20), in general, does not determine a system of conservation laws but impose additional constraints. Thus, Noether symmetries with arbitrary functions of time instead of conservation laws lead to a set of additional constraints imposed on the function u and its derivatives. Therefore, the satisfaction of the strict boundary conditions (15), along with the Noether boundary conditions (13), becomes critical in the sense of avoiding additional constraints (20). Correspondingly, we have three possible situations.

(1) Strict boundary conditions (15), along with the Noether boundary conditions (13), can be satisfied for an arbitrary function $\gamma(t)$. Then the system (20) instead of conservation laws provides additional constraints that the function u and its derivatives must satisfy.

(2) Strict boundary conditions (15) along with the Noether boundary conditions (13), can be satisfied for some particular functions $\gamma(t)$. In this case the (finite) symmetry (16) leads to the Noether conservation law (17) in agreement with the first Noether theorem.

(3) Strict boundary conditions (15) cannot be satisfied for any function $\gamma(t)$. In this case a consequence of an infinite symmetry (16) will be that the solutions of the original differential equation with the boundary conditions (13) and (15) do not exist.

Thus, in order to avoid additional constraints (20) we have to find those particular functions $\gamma(t)$ that lead to different boundary conditions than the ones in general case, when the function $\gamma(t)$ is arbitrary (Rosenhaus 2002). Each choice of such functions $\gamma(t)$ gives rise to a respective conserved quantity.

3. Symmetries and conservation laws for the Davey–Stewartson equations

Let us apply the above approach for finding non-vanishing conserved densities of the DS equations with boundary conditions at infinity. We study conserved densities of the DS equations:

$$\begin{aligned} &\mathrm{i}\psi_t + \psi_{xx} + \epsilon\psi_{yy} - k\psi|\psi|^2 - \mu\psi\phi_y = 0, \\ &\phi_{xx} - \epsilon\phi_{yy} + \mu(|\psi|^2)_y = 0, \end{aligned} \tag{21}$$

where $\psi = \psi(x, y, t)$ is a complex-valued function, $\phi = \phi(x, y, t)$ is a real-valued function, k, μ are real constants and $\epsilon = \pm 1$. The function $\psi(x, y, t)$ is the amplitude of the water wave and $\phi(x, y, t)$ is related to the wave mean velocity potential. The case $\epsilon = 1$ corresponds to the DSI equations and $\epsilon = -1$ leads to the DSII equation, see e.g. Kaup (1993). As was mentioned above, the DS system can be considered as two-dimensional generalization of a nonlinear Schrödinger equation for a complex-valued field $\psi = \psi(x, y, t)$ with the self-interaction $|\psi^4|$ and coupling with a scalar field $\phi(x, y, t)$. The corresponding Lagrangian is

$$L = \frac{\mathrm{i}\hbar}{2}(\psi^*\psi_t - \psi_t^*\psi) - \frac{\hbar^2}{2m}(\psi_x^*\psi_x + \psi_y^*\psi_y) - \frac{1}{2}\left(\phi_x^2 - \phi_y^2\right) - \frac{k}{2}|\psi|^4 - \mu\phi_y|\psi|^2.$$

In terms of real functions $u = u(x, y, t)$, $v = v(x, y, t)$

$$\psi = u + \mathrm{i}v, \qquad \psi^* = u - \mathrm{i}v. \tag{22}$$

The DS equations (21) have a form

$$\begin{aligned} &-v_t + u_{xx} + \epsilon u_{yy} - ku(u^2 + v^2) - \mu u\phi_y = 0, \\ &u_t + v_{xx} + \epsilon v_{yy} - kv(u^2 + v^2) - \mu v\phi_y = 0, \\ &\phi_{xx} - \epsilon\phi_{yy} + \mu(u^2 + v^2)_y = 0, \end{aligned} \tag{23}$$

where $u, v, \phi \subset C^2$. The Lagrangian for the system (23) is (Omote 1988, Leble *et al* 1992) (compare with the L above for $\hbar = m = 1$)

$$L = vu_t - uv_t - \left(u_x^2 + v_x^2 + \epsilon u_y^2 + \epsilon v_y^2\right) - \frac{1}{2}\left(\phi_x^2 - \epsilon\phi_y^2\right) - \frac{k}{2}(u^2 + v^2)^2 - \mu\phi_y(u^2 + v^2). \tag{24}$$

The following operators determine the Lie point symmetry group of equations (23) (compare with Champagne and Winternitz (1988) and Omote (1988) for $\epsilon = \pm 1$):

$$
\begin{aligned}
X_g &= g\frac{\partial}{\partial x} - \frac{x}{2}g'\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right) - \frac{xy}{2\mu}g''\frac{\partial}{\partial \phi},\\
X_l &= l\frac{\partial}{\partial y} - \frac{y}{2\epsilon}l'\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right) - \frac{x^2 + y^2/\epsilon}{4\mu}l''\frac{\partial}{\partial \phi},\\
X_f &= f\frac{\partial}{\partial t} + \frac{f'}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - \phi\frac{\partial}{\partial \phi}\right)\\
&-\frac{f''}{8}(x^2 + y^2/\epsilon)\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right) - \frac{f'''}{8\mu}(x^2y + y^3/3\epsilon)\frac{\partial}{\partial \phi},\\
X_h &= h\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right) + \frac{yh'}{\mu}\frac{\partial}{\partial \phi},\\
X_\alpha &= (x\alpha/\mu)\frac{\partial}{\partial \phi}, \qquad X_\beta = (\beta/\mu)\frac{\partial}{\partial \phi},
\end{aligned}
\tag{25}
$$

where $g(t), l(t), f(t), h(t), \alpha(t), \beta(t)$ are arbitrary smooth functions. Let us analyze the conserved quantities corresponding to infinite subalgebras of algebra (25). First, we write our symmetry operators in canonical form. For an operator

$$
X = \xi^t\frac{\partial}{\partial t} + \xi^x\frac{\partial}{\partial x} + \xi^y\frac{\partial}{\partial y} + \cdots + \eta^a\frac{\partial}{\partial u^a}, \qquad a = 1, \ldots, n, \tag{26}
$$

a corresponding canonical operator takes a form

$$
X_\alpha = X - \xi^i D_i = \alpha^a\frac{\partial}{\partial u^a} + \zeta_i^a\frac{\partial}{\partial u_i^a} + \sigma_{ij}^a\frac{\partial}{\partial u_{ij}^a}\ldots, \tag{27}
$$

where

$$
\alpha^a = \eta^a - \xi^i u_i^a, \qquad \zeta_i^a = D_i\alpha^a, \qquad \sigma_{ij}^a = D_{ij}\alpha^a, \qquad a = 1, \ldots, n. \tag{28}
$$

We will start with the symmetry operator X_g and find corresponding conserved densities (Rosenhaus 2003, 2005).

4. Essential conservation laws associated with X_g

We have in our case

$$
X_\alpha = \alpha^u\frac{\partial}{\partial u} + \alpha^v\frac{\partial}{\partial v} + \alpha^\phi\frac{\partial}{\partial \phi} + (D_i\alpha^u)\frac{\partial}{\partial u_i} + (D_i\alpha^v)\frac{\partial}{\partial v_i} + (D_i\alpha^\phi)\frac{\partial}{\partial \phi_i}, \quad i = x, y, t. \tag{29}
$$

Using (28), (26) and (16) we see that

$$
\begin{aligned}
&\xi^x = g, \quad \xi^t = \xi^y = 0, \quad \eta^u = -xg'v/2, \quad \eta^v = xg'u/2, \quad \eta^\phi = -xyg''/2\mu,\\
&\alpha^u = -gu_x - xg'v/2, \quad \alpha^v = -gv_x + xg'u/2, \quad \alpha^\phi = -g\phi_x - xyg''/2\mu.
\end{aligned}
\tag{30}
$$

Calculating $X_\alpha L$ we obtain

$$
X_\alpha L = D_x(-gL + g''y\phi/2\mu) + D_y(-\epsilon g''x\phi/2\mu). \tag{31}
$$

Thus, X_g is a Noether symmetry operator and using (6) and (18) we can write

$$
X_\alpha L = D_i M_i, \qquad M_x = -gL + g''y\phi/2\mu, \quad M_y = -\epsilon g''x\phi/2\mu, \quad M_t = 0. \tag{32}
$$

The form of Noether and strict boundary conditions depends on the function $g(t)$.

(A) *$g(t)$ is arbitrary*. Noether boundary conditions (13) for X_α are

$$
L, \phi \underset{x\to\pm\infty}{\rightarrow} 0, \qquad \phi \underset{y\to\pm\infty}{\rightarrow} 0, \tag{33}
$$

which can be specified as

$$u, v, u_x, u_y, v_x, v_y, \phi, \phi_x, \phi_y \underset{x\to\pm\infty}{\rightarrow} 0, \qquad \phi \underset{y\to\pm\infty}{\rightarrow} 0. \tag{34}$$

Strict boundary conditions (15) take the form

$$\begin{aligned} &\alpha^u L_{u_x} + \alpha^v L_{v_x} + \alpha^\phi L_{\phi_x}\big|_{x\to\pm\infty} = 0, \\ &\alpha^u L_{u_y} + \alpha^v L_{v_y} + \alpha^\phi L_{\phi_y}\big|_{y\to\pm\infty} = 0, \end{aligned} \tag{35}$$

or

$$\begin{aligned} &g(2{u_x}^2 + 2{v_x}^2 + {\phi_x}^2) + g'x(vu_x - uv_x) + g''xy\phi_x/2\mu \underset{x\to\pm\infty}{\rightarrow} 0, \\ &g(2\epsilon u_x u_y + 2\epsilon v_x v_y - \phi_x\phi_y + \mu(u^2+v^2)\phi_x) \\ &\qquad + \epsilon g'x(vu_y - uv_y) - g''xy(\epsilon\phi_y/\mu - (u^2+v^2)/2) \underset{y\to\pm\infty}{\rightarrow} 0. \end{aligned} \tag{36}$$

Thus, for an arbitrary function $g(t)$ the strict boundary conditions can be specified as

$$\begin{aligned} &u_x, v_x, x\phi_x, x(vu_x - uv_x) \underset{x\to\pm\infty}{\rightarrow} 0, \\ &yu^2, yv^2, u_x u_y, v_x v_y, y\phi_y, (vu_y - uv_y) \underset{y\to\pm\infty}{\rightarrow} 0. \end{aligned} \tag{37}$$

The symmetry transformation X_g for arbitrary $g(t)$ leads to a system of additional constraints (20) instead of conservation laws. In order to avoid restrictions (20), let us consider some specific forms of $g(t)$ for which we can weaken our boundary conditions (34) and (37).

(B) $g'(t) = 0,\ g(t) = \text{const}$. In this case α and M_i simplify to

$$\alpha^u = -u_x, \qquad \alpha^v = -v_x, \qquad \alpha^\phi = -\phi_x, \qquad M_x = -L, \quad M_y = M_t = 0. \tag{38}$$

Noether boundary conditions are

$$u, v, uv_t - vu_t, u_x, u_y, v_x, v_y, \phi_x, \phi_y \underset{x\to\pm\infty}{\rightarrow} 0. \tag{39}$$

For strict boundary conditions, in addition to (39) we have

$$\phi_x \underset{x\to\pm\infty}{\rightarrow} 0, \qquad u_x u_y, v_x v_y, \phi_x\phi_y, \phi_x(u^2+v^2) \underset{y\to\pm\infty}{\rightarrow} 0. \tag{40}$$

Since the boundary conditions (39) and (40) for this case are less restrictive than corresponding boundary conditions in the case of an arbitrary function $g(t)$, then according to (17) a symmetry X_g will lead to the following essential conservation law:

$$D_t \iint (uv_x - vu_x)\,\mathrm{d}x\,\mathrm{d}y \doteq 0, \tag{41}$$

or

$$P_x = \iint (uv_x - vu_x)\,\mathrm{d}x\,\mathrm{d}y \doteq \text{const}.$$

Expression (41) is conservation of the x-component of linear momentum P_x of the system that takes place when boundary conditions (39) and (40) are satisfied,

$$P_x \equiv \iint p_x\,\mathrm{d}x\,\mathrm{d}y, \qquad p_x = uv_x - vu_x.$$

The corresponding continuity equation has the form

$$\begin{aligned} &D_x\left[L + 2\left(u_x^2 + v_x^2\right) + {\phi_x}^2\right] + D_y\left[2\epsilon(u_x u_y + v_x v_y) - \epsilon\phi_x\phi_y + \mu\phi_x(u^2+v^2)\right] - \\ &D_t(v_x u - u_x v) \doteq 0. \end{aligned} \tag{42}$$

(C) $g''(t) = 0,\ g'(t) \neq 0$: $g(t) = t$. We have

$$M_x = -tL, \qquad M_y = M_t = 0. \tag{43}$$

Noether boundary conditions are the same as in case B (39). For strict boundary conditions in addition to (40) we get

$$x(uv_x - vu_x) \underset{x\to\pm\infty}{\to} 0, \qquad uv_y - vu_y \underset{y\to\pm\infty}{\to} 0. \tag{44}$$

The essential conservation law associated with the boundary conditions (39), (40) and (44) takes the form

$$D_t \iint [2t(uv_x - vu_x) - x(u^2 + v^2)]\, \mathrm{d}x\, \mathrm{d}y \doteq 0. \tag{45}$$

Note that the conservation (45) implies that

$$D_t \iint x(u^2 + v^2)\, \mathrm{d}x\, \mathrm{d}y \doteq 2P_x, \tag{46}$$

and $\iint x(u^2 + v^2)\, \mathrm{d}x\, \mathrm{d}y \doteq$ const only when $P_x \doteq 0$.

(D) $g''(t) \neq 0$ In this case, Noether and strict boundary conditions have the same form (34) and (37) as in case A and lead to no essential conservation laws.

5. Essential conservation laws associated with X_l

For a corresponding canonical operator X_α (29) we get

$$\alpha^u = -lu_y - yl'v/2\epsilon, \qquad \alpha^v = -lv_y + yl'u/2\epsilon, \qquad \alpha^\phi = -l\phi_y - (x^2 + y^2/\epsilon)l''/4\mu. \tag{47}$$

Calculating $X_\alpha L$ we obtain

$$X_\alpha L = D_x(x\phi l''/2\mu) - D_y(lL + y\phi l''/2\mu). \tag{48}$$

Thus, X_l is a Noether symmetry operator and using (6) and (17)

$$X_\alpha L = D_i M_i, \qquad M_x = x\phi l''/2\mu, \quad M_y = -lL - y\phi l''/2\mu, \quad M_t = 0. \tag{49}$$

As in the previous case the form of strict and Noether boundary conditions depends on the function $l(t)$.

(A) $l(t)$ is arbitrary. From the Noether and strict boundary conditions we get

$$uv_x - vu_x, u_x u_y, v_x v_y, x\phi, x^2\phi_x \underset{x\to\pm\infty}{\to} 0,$$
$$yu, yv, u_x, u_y, v_x, v_y, uv_t - vu_t, y\phi, y^2\phi_y\, \phi_x \underset{y\to\pm\infty}{\to} 0. \tag{50}$$

No local conservation laws are associated with the Noether transformation X_l when $l(t)$ is arbitrary. Let us consider now some specific forms of the function $l(t)$ for which we can weaken boundary conditions (50).

(B) $l' = 0, \quad l(t) = 1$. We have

$$\alpha^u = -u_y, \quad \alpha^v = -v_y, \quad \alpha^\phi = -\phi_y, \qquad M_x = M_t = 0, \qquad M_y = -L. \tag{51}$$

The Noether boundary conditions look as follows:

$$u, v, u_x, u_y, v_x, v_y, uv_t - vu_t, \phi_x, \phi_y \underset{y\to\pm\infty}{\to} 0. \tag{52}$$

The strict boundary conditions, in addition to (52), require that

$$u_x u_y + v_x v_y + \phi_x \phi_y/2 \underset{x\to\pm\infty}{\to} 0. \tag{53}$$

Boundary conditions (52) and (53) are softer than those for the case of an arbitrary function $l(t)$ (50) and according to (17), the symmetry operator X_l leads to the following conservation law:

$$D_t \iint (uv_y - vu_y)\,\mathrm{d}x\,\mathrm{d}y \doteq 0. \tag{54}$$

Expression (54) is a conservation of the y-component of linear momentum of the system P_y with regular boundary conditions (52) and (53).

$$P_y = \iint (uv_y - vu_y)\,\mathrm{d}x\,\mathrm{d}y \doteq \mathrm{const}.$$

(C) $l'' = 0,\ \ l' \neq 0:\ l(t) = t$. We have

$$M_y = -tL, \qquad M_x = M_t = 0. \tag{55}$$

The Noether boundary conditions are the same as in case B (52). For strict boundary conditions in addition to (52) and (53) we get

$$uv_x - vu_x \underset{x\to\pm\infty}{\to} 0, \qquad y(uv_y - vu_y) \underset{y\to\pm\infty}{\to} 0. \tag{56}$$

The essential conservation law associated with the boundary conditions (52), (53) and (56) takes the form

$$D_t \iint [2t(uv_y - vu_y) - y(u^2 + v^2)]\,\mathrm{d}x\,\mathrm{d}y \doteq 0. \tag{57}$$

From expression (57) we can see that

$$D_t \iint (y(u^2 + v^2)\,\mathrm{d}x\,\mathrm{d}y \doteq 2P_y.$$

(D) $l''(t) \neq 0$ In this case, Noether and strict boundary conditions have the same form (50) as in case A and lead to no essential conservation laws.

6. Essential conservation laws associated with X_f

For X_α in this case we have

$$\begin{aligned} \alpha^u &= fu_t + f'(xu_x + yu_y + u)/2 + f''(x^2 + y^2/\epsilon)v/8, \\ \alpha^v &= fu_t + f'(xv_x + yv_y + v)/2 - f''(x^2 + y^2/\epsilon)u/8, \\ \alpha^\phi &= f\phi_t + f'(x\phi_x + y\phi_y + \phi)/2 + f'''(x^2y + y^3/3\epsilon)/8\mu. \end{aligned} \tag{58}$$

Calculating $X_\alpha L$ we obtain

$$\begin{aligned} X_\alpha L &= D_i M_i, \qquad & M_x &= xLf'/2 - \epsilon xy\phi f'''/4\mu, \\ M_t &= fL, \qquad & M_y &= yLf'/2 + (\epsilon x^2 + y^2)\phi f'''/8\mu. \end{aligned} \tag{59}$$

As in the previous cases the form of strict and Noether boundary conditions depends on the function $f(t)$.

(A) $f(t)$ is arbitrary. From the Noether and strict boundary conditions we will get

$$x^2(v_xu - u_xv),\, xv_tu,\, xu_tv,\, xu_x^2,\, xu_y^2,\, xv_x^2,\, xv_y^2,\, xu^4,\, xv^4,\, x\phi,\, x^2\phi_x,\, x\phi_y^2 \underset{x\to\pm\infty}{\to} 0,$$

$$y^2(v_yu - u_yv),\, yv_tu,\, yu_tv,\, yu_x^2,\, yu_y^2,\, yv_x^2,\, yv_y^2,\, yu^4,\, yv^4,\, y^2\phi,\, y^3\phi_y\, y\phi_y^2 \underset{y\to\pm\infty}{\to} 0,$$

$$f(t)L \underset{t\to\pm\infty}{\to} 0, \qquad \forall f(t). \tag{60}$$

No local conservation laws are associated with the Noether transformation X_α with an arbitrary function $f(t)$. Let us consider some specific forms of $f(t)$ for which we can weaken boundary conditions (60).

(B) $f' = 0, \quad f(t) = 1$. We have

$$\alpha^u = u_t, \quad \alpha^v = v_t, \quad \alpha^\phi = \phi_t, \qquad M_x = M_y = 0, \qquad M_t = L. \tag{61}$$

The Noether and strict boundary conditions here look as follows:

$$u_x u_t + v_x v_t + \phi_x \phi_t \underset{x\to\pm\infty}{\rightarrow} 0, \qquad u_y u_t + v_y v_t - \phi_y \phi_t \underset{y\to\pm\infty}{\rightarrow} 0, \qquad L \underset{t\to\pm\infty}{\rightarrow} 0. \tag{62}$$

Boundary conditions (62) are considerably softer than in the case of an arbitrary function $f(t)$ (60) and according to (17), the symmetry operator X_f leads to the following associated conservation law:

$$D_t \iint \left[\left(u_x^2 + v_x^2 + \epsilon u_y^2 + \epsilon v_y^2\right) + \frac{\phi_x^2 - \epsilon \phi_y^2}{2} + \frac{k}{2}(u^2+v^2)^2 + \mu \phi_y (u^2+v^2) \right] \mathrm{d}x\,\mathrm{d}y \doteq 0. \tag{63}$$

Integrand in (63) is the density of the Hamiltonian H of the system, and as we can see energy conservation law (63) of the system requires rather soft boundary conditions (62).

(C) $f'' = 0, \;\; f' \neq 0: \; f(t) = 2t$. We have

$$M_x = xL, \qquad M_y = yl, \qquad M_t = 2tL. \tag{64}$$

The Noether and strict boundary conditions here require that, in addition to (62),

$$x(v_t u - u_t v), xu_x^2, xu_y^2, xv_x^2, xv_y^2, xu^4, xv^4, x\phi_x^2, x\phi_y^2 \underset{x\to\pm\infty}{\rightarrow} 0,$$
$$y(v_t u - u_t v), yu_x^2, yu_y^2, yv_x^2, yv_y^2, yu^4, yv^4, y\phi_x^2, y\phi_y^2 \underset{y\to\pm\infty}{\rightarrow} 0. \tag{65}$$

The essential conservation law associated with the symmetry operator X_f and boundary conditions (65) takes the form

$$D_t \iint [2tH - x(uv_x - vu_x) - y(uv_y - vu_y)]\,\mathrm{d}x\,\mathrm{d}y \doteq 0 \tag{66}$$

or

$$D_t \iint [2tH - xp_x - yp_y]\,\mathrm{d}x\,\mathrm{d}y \doteq 0. \tag{67}$$

It follows from expression (67) that

$$D_t \iint \left[xp_x + yp_y\right] \mathrm{d}x\,\mathrm{d}y \doteq 2 \iint H\,\mathrm{d}x\,\mathrm{d}y. \tag{68}$$

(D) $f''' = 0, \;\; f'' \neq 0: \quad f(t) = t^2$. We have

$$M_x = xtL, \qquad M_y = ytL, \qquad M_t = t^2 L. \tag{69}$$

The corresponding essential conserved quantity here is, according to (17),

$$D_t \iint \left[t^2 H - t(xp_x + yp_y) + (x^2 + y^2/\epsilon)(u^2+v^2)/4\right] \mathrm{d}x\,\mathrm{d}y \doteq 0 \tag{70}$$

(compare with Ozawa (1992)), with the following boundary conditions:

$$x^2(v_x u - u_x v), xv_t u, xu_t v, xu_x^2, xu_y^2, xv_x^2, xv_y^2, xu^4, xv^4, x\phi_x^2, x\phi_y^2 \underset{x\to\pm\infty}{\rightarrow} 0,$$
$$y^2(v_y u - u_y v), yv_t u, yu_t v, yu_x^2, yu_y^2, yv_x^2, yv_y^2, yu^4, yv^4, y\phi_x^2, y\phi_y^2 \underset{y\to\pm\infty}{\rightarrow} 0, \tag{71}$$
$$t^2 L \underset{t\to\pm\infty}{\rightarrow} 0.$$

Taking into consideration (68) we obtain

$$D_t^2 \iint [(x^2 + y^2/\epsilon)(u^2+v^2)]\,\mathrm{d}x\,\mathrm{d}y \doteq 2 \iint H\,\mathrm{d}x\,\mathrm{d}y, \tag{72}$$

see Ablowitz and Segur (1979) and Ghidaglia and Saut (1990).

(E) $f''' \neq 0$. Noether and strict boundary conditions in this case have the same form (60) as for the general case, and the symmetry X_f leads to no essential conservation laws.

7. Essential conservation laws associated with X_h

We get

$$\alpha^u = hv, \qquad \alpha^v = -hu, \qquad \alpha^\phi = yh'/\mu, \qquad (73)$$
$$M_x = 0, \qquad M_y = \epsilon h'\phi/\mu, \qquad M_t = 0.$$

The form of strict and Noether boundary conditions depends on the function $h(t)$.

(A) $h(t)$ is arbitrary. The Noether and strict boundary conditions have a form

$$uv_x - vu_x, \phi_x \underset{x\to\pm\infty}{\to} 0, \qquad (74)$$
$$uv_y - vu_y, yu^2, yv^2, \phi, y\phi_y \underset{y\to\pm\infty}{\to} 0.$$

No local conservation laws are associated with the Noether transformation X_h when $h(t)$ is arbitrary.

(B) $h' = 0, \quad h(t) = 1$. We have

$$\alpha^u = v, \qquad \alpha^v = -u, \qquad \alpha^\phi = 0, \qquad M_x = M_y = M_t = 0. \qquad (75)$$

The boundary conditions in this case look as follows:

$$uv_x - vu_x \underset{x\to\pm\infty}{\to} 0, \qquad uv_y - vu_y \underset{y\to\pm\infty}{\to} 0. \qquad (76)$$

Boundary conditions (76) are weaker than those in the case of an arbitrary function $h(t)$ (74) and the symmetry operator X_l leads to the following essential conservation law:

$$2D_xP_x + 2D_yP_y + D_t(u^2+v^2) \doteq 0 \qquad (77)$$

or

$$D_t \iint (u^2+v^2)\,\mathrm{d}x\,\mathrm{d}y \doteq 0, \qquad (78)$$

showing the conservation of mass (number of particles) of the system (e.g. Ozawa (1992)).

(C) $h' \neq 0$. Noether and strict boundary conditions are the same as in the case of an arbitrary function $h(t)$ (74). In this case, our symmetry X_h does not lead to essential conservation laws.

Infinite symmetries X_α and X_β do not lead to any essential conservation laws.

8. Essential symmetries

Let us discuss the symmetry transformations that give rise to essential conservation laws.

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial t},$$
$$X_4 = t\frac{\partial}{\partial x} - \frac{x}{2}\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right),$$
$$X_5 = t\frac{\partial}{\partial y} - \frac{\epsilon y}{2}\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right),$$
$$X_6 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \qquad (79)$$
$$X_7 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - \phi\frac{\partial}{\partial \phi},$$
$$X_8 = t^2\frac{\partial}{\partial t} + t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - \phi\frac{\partial}{\partial \phi}\right) - \frac{x^2+\epsilon y^2}{4}\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right).$$

We can see that our 'essential' symmetry transformations include translations (with generators X_1, X_2, X_3), extended Galilean transformations (X_4 and X_5), rotations in u, v space (X_6), dilatation in extended space x, y, t, u, v, ϕ (X_7) and extended projective transformation (X_8). The set of operators $X_1, X_2, \ldots, X_8$ (79) forms an algebra with the following non-vanishing commutation relations:

$$
\begin{aligned}
&[X_1, X_4] = -X_6/2, \qquad &&[X_2, X_5] = -\epsilon X_6/2, \qquad &&[X_3, X_4] = X_1,\\
&[X_3, X_5] = X_2, &&[X_1, X_7] = X_1, &&[X_2, X_7] = X_2,\\
&[X_4, X_7] = -X_4, &&[X_5, X_7] = -X_5, &&[X_3, X_8] = X_7,\\
&[X_3, X_7] = 2X_3, &&[X_3, X_8] = X_7.
\end{aligned}
$$

Note that algebra (79) is not a (minimal) set of operators generating algebra (25) (see Ibragimov (1985)). Out of all operators of an infinite algebra (25), only symmetries (79) lead to nonzero local conserved quantities. Note also that each essential conservation law determined by operators (79) corresponds to a specific boundary condition.

9. Conclusions

In the paper we have generated a set of essential conservation laws for the system of Davey–Stewartson (DS) equations (23), associated with its classical Lie point symmetry group. Out of infinitely many continuity equations for the DS system corresponding to its infinite symmetry algebra (containing six arbitrary functions) we identified those that lead to non-vanishing conserved quantities. Each of eight conserved quantities we obtained ((41), (45), (54), (57), (63), (66), (70) and (78)) corresponds to a special form of one of the arbitrary functions in the generators of the symmetry group, that require softer boundary conditions than in the general case. Thus, each of our essential conservation laws is determined by a specific form of boundary conditions, and known conservation laws of linear momentum, energy and mass (41), (54), (63) and (78) correspond to the weakest boundary conditions. Other essential local conservation laws assume stricter asymptotic behavior.

Acknowledgment

VR would like to thank Vladimir Gerdjikov for useful discussion.

References

Ablowitz M J and Segur H 1979 *J. Fluid Mech.* **92** 592–691
Anker D and Freeman N C 1978 *Proc. R. Soc.* A **360** 592
Benney D J and Roskes G J 1969 *Stud. Appl. Math.* **48** 377
Boiti M, Leon J J-P, Martina L and Pempinelli F 1988 *Phys. Lett.* **132A** 432
Champagne B and Winternitz P 1988 *J. Math. Phys.* **29** 1
Davey A and Stewartson K 1974 *Proc. R. Soc.* A **338** 101
Djordjevic V D and Redecopp L G 1977 *J. Fluid Mech.* **79** 103
Fokas A S and Ablowitz M J 1984 *J. Math. Phys.* **25** 2494
Fokas A S and Santini P M 1990 *Physica* D **44** 99
Fokas A S and Santini P M 1988a *Symmetries and Nonlinear Phenomena* (Teaneck, NJ: World Scientific) pp 7–21
Fokas A S and Santini P M 1988b *Commun. Math. Phys.* **116** 449
Ibragimov N H 1985 *Transformation Groups Applied to Mathematical Physics* (Boston, MA: Reidel)
Ghidaglia J-M and Saut J-C 1990 *Nonlinearity* **3** 475
Kaup D J 1993 *Inverse Problems* **9** 417
Kulish P P and Lipovsky V D 1987 *Zapiski Nauchnogo Seminara LOMI* **161** 54–91

Leble S B, Salle M A and Yurov A V 1992 *Inverse Problems* **8** 207
Lou S and Hu X 1994 *Inverse Problems* **27** L207
Nakamura 1982 *J. Math. Phys.* **23** 1422
Noether E 1918 *Nachr. Koönig. Gessell. Wissen. Goöttingen, Math.-Phys. Kl. Heft* **2** 235
Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)
Omote M 1988 *J. Math. Phys.* **29** 2599
Ozawa T 1992 *Proc. Roy. Soc.* A **436** 345
Pang G-D and Zhao B-H 1993 *J. Math. Phys.* **34** 1063
Rosen J 1972 *Some Properties of the Euler–Lagrange Operators* (Tel-Aviv: Tel-Aviv University) TAUP-269-72
Rosenhaus V 2002 *J. Math. Phys.* **43** 6129
Rosenhaus V 2003 *Rep. Math. Phys.* **51** 71
Rosenhaus V 2005 *Theor. Math. Phys.* **144** 1046
Rosenhaus V 2006 *J. Phys. A: Math. Gen.* **39** 7693
Sulem C and Sulem P-L 1999 *The Nonlinear Schrödinger Equation. Self-focusing and Wave Collapse (Applied Mathematical Sciences* vol 139*)* (New York: Springer)
Zakharov V E and Shabat A B 1973 *Zh. Eksp. Teor. Fiz.* **37** 823